Energy-levels crossing and radial Dirac equation: Supersymmetry and quasi-parity spectral signatures

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Abstract

The (3+1)-dimensional Dirac equation with position dependent mass in 4-vector electromagnetic fields is considered. Using two over-simplified examples (the Dirac-Coulomb and Dirac-oscillator fields), we report energy-levels crossing as a spectral property or as an effect of the hidden supersymmetric quantum mechanical language and/or quasi-parity signatures. Under different settings of the related interactions' way-of-coupling into Dirac equation, it is observed that the two ultimate/effective descendents, Dirac-Coulomb and Dirac-oscillator, exhibit different conditions on the energy-levels crossings.

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1 Introduction

The search for exact-solvability for quantum mechanical systems, both non-relativistic (Schrödinger equation) and relativistic (e.g., Klein-Gordon and Dirac equations), is inviting and desirable. Exactly-solvable quantum mechanical systems are vital ingredients for (among others that are mathematically motivated, say) the conceptual understanding of physics and for inspiring the structure of the numerical methods designed to solve more complicated physical problems. However, whilst intensive attention was paid to exact-solvability of the non-relativistic Schrödinger equation, the relativistic Klein-Gordon and Dirac equations remained unfortunate and only partially attended in the yet already partially-explored Dirac territories.

Some exactly-solvable potentials, for example, are known to belong to some distinctive classes of shape invariant potentials [1]. Within each class of which the so-called point-canonical-transformation (PCT) [2] would map the solution

(eigenvalues and eigenfunctions) of one into another. On the other hand, supersymmetric quantum mechanics [3] and potential algebras [4] (among others of course) are known to be used to obtain exact solutions for quantum mechanical systems. Yet, in between exactly-solvable and non-exactly-solvable there exists the gray zone of conditionally-exactly-solvable (i.e., all the spectrum is obtained) [5] and quasi-exactly-solvable (i.e., part of the spectrum is obtained) [6] potentials models.

Nevertheless, we may recollect that the supersymmetric quantum mechanical language is realized as a *hidden/built-in symmetry* in the (1+1)-dimensional Dirac equation (cf., e.g. [3,7–9]). Nogami and Toyama [7] have reported that the associated Schrödinger supersymmetric-partner Hamiltonians share the same energy spectrum including the lowest states unless Dirac equation allows a zero-mode (i.e., zero-energy bound-state). Jackiw and Rebbi [9] have, moreover, reported that only for some certain *topological* trends where the Lorentz scalar potential $S(x)$ is localized (i.e., $S(x) \rightarrow 0$ for $x \rightarrow \pm\infty$), Dirac equation would allow a zero-mode.

One should be reminded, hereupon, that a Lorentz scalar potential (and/or an almost mathematically and partially-physically equivalent position-dependent mass) in Dirac equation is mainly motivated by the MIT bag model of quarks (cf., e.g., [10] and references therein). Yet, a Lorentz scalar potential couples to the mass of the fermion instead of its charge and the related positive and negative energies exhibit identical behaviors. Moreover, the position-dependent mass settings are useful models to study, for example, the energy density many-body problem, electronic properties of semiconductors and quantum dots, etc (cf., e.g., sample of references in [11-18]). We may also recollect that the phenomenon of *energy-levels crossing* is responsible for electron transfer in protein, it underlies stability analysis in mechanical engineering, and mathematically appears in algebraic geometry (cf., e.g., [19] and related references therein).

Very recently [20], we have reemphasized the hidden/built-in supersymmetric quantum mechanical language in the spectrum of the (1+1)-dimensional Dirac equation with position-dependent mass and complexified Lorentz scalar interactions. We have reported the "quasi-parity" signature on the Dirac spectrum and discussed energy-levels crossings related to supersymmetry and/or "quasi-parity" signatures. We have observed that the supersymmetric signature on the (1+1)-Dirac spectrum is documented through the emergence of "exact" isospectral (i.e., including the lowest-state) partner Hamiltonians for "even"-quasi-parity, whereas the partner Hamiltonians share the same energy spectrum with a "missing" lowest-state for "odd"-quasi-parity, at least for the examples discussed therein. It would be interesting, we contemplate, if such studies are extended to cover the (3+1)-dimensional radial Dirac equation with different models of interactions' couplings and with position-dependent mass settings. Such studies merely exist in the literature, to the best of our knowledge, and may very well add a new flavour to the readily "multi-flavoured" Dirac equation.

This article is organized as follows. In section 2, we recollect the (3+1)-dimensional radial Dirac equation with position-dependent mass in a four-vector

electromagnetic field. Therein, we realize that the decoupled one-dimensional Schrödinger-like radial Dirac equations exhibit supersymmetric language only when $M(r) = V(r) = 0$, where $M(r) = m(r) + S(r)$ with the position-dependent mass $m(r)$, the Lorentz scalar potential $S(r)$, and the Lorentz vector potential $V(r)$. In section 3, the consequences of an equally-mixed non-zero Lorentz vector and Lorentz scalar potential settings are discussed through illustrative examples: Dirac-Coulomb-I, Dirac-oscillator-I, and Dirac-oscillator-II. The consequences of $V(r) \neq M(r)$ with the magnetic interaction $A(r) = -\zeta_2'(r)/[2\zeta_2(r)] \neq 0$ (see Eq.(7) below) are given in section 4, along with illustrative examples: Dirac/Klein-Gordon-Coulomb-II and Dirac/Klein-Gordon-oscillator-III. In section 5, we report the consequences of an equally-mixed Lorentz vector and scalar *"free"*-fields (i.e., $M(r) = V(r) = 0$). The supersymmetric quantum mechanical language and the "quasi-parity" signatures on the spectra of a *Dirac-oscillator-toy* and a *Dirac-Coulomb-toy* models are reported in the same section. We conclude in section 6.

2 Radial Dirac equation with position dependent mass in a 4-vector electromagnetic field, recollected

The Hamiltonian describing a Dirac particle (in $\hbar = c = e = 1$ units) in a four-vector electromagnetic field $A_\mu = \left(A_0, \vec{A}\right) = \left(V, \vec{A}\right)$ reads

$$H_D = \overrightarrow{\alpha} \cdot \left(\overrightarrow{p} - i\overrightarrow{A}\right) + \beta m + V, \tag{1}$$

with

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where σ_j are Pauli's 2×2 matrices and 1 is the 2×2 unit matrix. Under spherically symmetric settings, $\vec{A} \rightarrow \hat{r}A(r)$, $V \rightarrow V(r)$ accompanied (for the convenience of the current study) by a position-dependent mass Lorentz scalar field, $m \rightarrow m + m(r) + S(r) = m + M(r)$, the two-component Dirac equation reads

$$\begin{pmatrix} m + M(r) + V(r) & \frac{\kappa}{r} + A(r) - \partial_r \\ \frac{\kappa}{r} + A(r) + \partial_r & -m - M(r) + V(r) \end{pmatrix} \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = E \begin{pmatrix} g(r) \\ f(r) \end{pmatrix} \tag{2}$$

with energy E, and κ in the centrifugal term is given by

$$\kappa = \begin{cases} -(\ell+1) & \text{for } j = \ell + 1/2 \\ \ell & \text{for } j = \ell - 1/2 \end{cases} \Longrightarrow \kappa(\kappa+1) = \ell(\ell+1), \tag{3}$$

where $\ell = 0, 1, 2, \cdots$ is the angular momentum quantum number. Equation (2) decouples into

$$\zeta_1(r)\; g(r) - \left(\frac{\kappa}{r} + A(r) - \partial_r\right) f(r) = 0 \tag{4}$$

$$\zeta_2(r)\; f(r) - \left(\frac{\kappa}{r} + A(r) + \partial_r\right) g(r) = 0 \tag{5}$$

where

$$\zeta_1(r) = (E - m) - V(r) - M(r), \tag{6}$$

$$\zeta_2(r) = (E + m) - V(r) + M(r). \tag{7}$$

Substituting $f(r)$ of (5) into (4) would, with

$$\tilde{A}(r) = \frac{\kappa}{r} + A(r), \tag{8}$$

imply

$$\left\{-\partial_r^2 + \tilde{A}(r)^2 - \tilde{A}'(r) + \frac{\zeta_2'(r)}{\zeta_2(r)}\left[\tilde{A}(r) + \partial_r\right] - \zeta_1(r)\,\zeta_2(r)\right\} g(r) = 0 \tag{9}$$

where primes denote derivatives with respect to r. Moreover, a substitution of the form

$$g(r) = \phi_2(r)\exp\left(-\frac{P_2(r)}{2}\right) \;\; ; \;\; P_2'(r) = \frac{V'(r) - M'(r)}{\zeta_2(r)} \tag{10}$$

would remove the first order derivative and result in a one-dimensional Schrödinger-like equation

$$\left\{-\partial_r^2 + \tilde{A}(r)^2 - \tilde{A}'(r) + U_2(r) - \zeta_1(r)\,\zeta_2(r)\right\} \phi_2(r) = 0 \tag{11}$$

where

$$U_2(r) = \frac{\zeta_2'(r)}{\zeta_2(r)}\tilde{A}(r) + \left[\frac{3}{4}\left(\frac{\zeta_2'(r)}{\zeta_2(r)}\right)^2 - \frac{1}{2}\frac{\zeta_2''(r)}{\zeta_2(r)}\right]. \tag{12}$$

Similarly, substituting $g(r)$ of (4) into (5) and taking

$$f(r) = \phi_1(r)\exp\left(-\frac{P_1(r)}{2}\right) \;\; ; \;\; P_1'(r) = \frac{V'(r) + M'(r)}{\zeta_1(r)}$$

would imply

$$\left\{-\partial_r^2 + \tilde{A}(r)^2 + \tilde{A}'(r) + U_1(r) - \zeta_1(r)\,\zeta_2(r)\right\} \phi_1(r) = 0 \tag{13}$$

where

$$U_1(r) = -\frac{\zeta_1'(r)}{\zeta_1(r)}\tilde{A}(r) + \left[\frac{3}{4}\left(\frac{\zeta_1'(r)}{\zeta_1(r)}\right)^2 - \frac{1}{2}\frac{\zeta_1''(r)}{\zeta_1(r)}\right]. \tag{14}$$

It should be noted that the decoupled radial Dirac one-dimensional Schrödinger-like equations, (11) and (13), possess natural (though hidden/built-in) supersymmetric quantum mechanical language only when $M(r) = V(r) = 0$. In the forthcoming experiments we shall be focusing on the upper component one-dimensional Schrödinger-like radial Dirac equation in (11) and (12).

3 Consequences of an equally-mixed Lorentz vector and scalar fields; $V(r) = M(r)$

An equally-mixed Lorentz vector and scalar fields, $V(r) = M(r)$, would imply $\zeta_2(r) = E + m$, $\zeta_1(r) = E - m - 2M(r)$. Consequently, (11) reduces to

$$\left\{ -\partial_r^2 + \tilde{A}(r)^2 - \tilde{A}'(r) + 2M(r)[E+m] \right\} \phi_2(r) = \left[E^2 - m^2\right] \phi_2(r), \quad (15)$$

where $\tilde{A}(r) = \frac{\kappa}{r} + A(r)$, as given in (8).

3.1 Dirac-Coulomb-I:

For $A(r) = a/r$ and $M(r) = b/r$ equation (15) reads

$$\left\{ -\partial_r^2 + \frac{\tilde{\kappa}(\tilde{\kappa}+1)}{r^2} + \frac{2b[E+m]}{r} \right\} \phi_2(r) = \left[E^2 - m^2\right] \phi_2(r), \quad (16)$$

where

$$\tilde{\kappa} = \kappa + a = \left\{ \begin{array}{lll} a + (j + 1/2) & \text{for} & \kappa = +(j+1/2) \\ a - (j + 1/2) & \text{for} & \kappa = -(j+1/2) \end{array} \right. .$$

Equation (16) admits exact solution that can be very well inferred from the well known radial Schrödinger-Coulomb problem to yield (with the radial quantum number $n_r = 0, 1, 2, \cdots$)

$$\left[E^2 - m^2\right] = -\frac{b^2 [E+m]^2}{\tilde{n}^2}; \ \tilde{n} = n_r + \tilde{\kappa} + 1 > 0, \quad (17)$$

which in turn implies

$$E = \frac{m\left(\tilde{n}^2 - b^2\right)}{\tilde{n}^2 + b^2}. \quad (18)$$

However, the fact that $\tilde{\kappa} = a \pm (j + 1/2)$ would manifest energy-levels crossings to obtain. That is, a state labeled by $\tilde{n}_1 = n_{r1} + a + j_1 + 3/2$ would cross with a state labeled by $\tilde{n}_2 = n_{r2} + a - j_2 + 1/2$ when

$$\frac{\tilde{n}_1^2 - b^2}{\tilde{n}_1^2 + b^2} = \frac{\tilde{n}_2^2 - b^2}{\tilde{n}_2^2 + b^2} \Longrightarrow \tilde{n}_2 = \tilde{n}_1 \Longrightarrow n_{r2} - n_{r1} = j_1 + j_2 + 1.$$

Moreover, one may wish to mind the consequences associated with a complexified coupling constant in $M(r)$, and hence in $V(r)$, in such a way that $V(r) = M(r) = b/r = -ib_\circ/r$ (e.g., simulating, say, the interaction of a point nucleus with an imaginary charge iZe and a particle of charge $-e$). In this case, $V(r) = M(r)$ is $\mathcal{PT}$-symmetrized [21] and

$$E_{\mathcal{PT}} = \frac{m\left(\tilde{n}^2 + b_\circ^2\right)}{\tilde{n}^2 - b_\circ^2}$$

Not only the energy spectrum $E_{\mathcal{PT}}$ follows similar energy-levels crossing scenario as that of (18), but also it suffers from the so called *flown-away* (cf., e.g., [21]) states that disappear from the spectrum when $\tilde{n} = |b_\circ|$.

3.2 Dirac-Oscillator-I:

For $A(r) = br/2$ and $M(r) = 0$ equation (15) implies

$$\left\{-\partial_r^2 + \frac{\ell(\ell+1)}{r^2} + \frac{b^2}{4}r^2 + \kappa b - \frac{b}{2}\right\}\phi_2(r) = \left[E^2 - m^2\right]\phi_2(r), \tag{19}$$

where $\kappa(\kappa+1) = \ell(\ell+1)$ for both $\kappa = -(\ell+1)\,; j = \ell + 1/2$ and $\kappa = \ell; j = \ell - 1/2$ is considered. This would result in

$$E^2 - m^2 = b(2n_r + \ell + 3/2) + \kappa b - \frac{b}{2}, \tag{20}$$

to imply

$$E_\pm = \pm\sqrt{m^2 + b(2n_r + \ell + \kappa + 1)}. \tag{21}$$

Obviously this result depends on the combination of the quantum numbers $2n_r + \ell = \Lambda$ and splits into

$$E_\pm = \left\{\begin{array}{lll} \pm\sqrt{m^2 + b(\Lambda + j + 3/2)} & \text{for} & \kappa = +(j+1/2) \\ \pm\sqrt{m^2 + b(\Lambda - j - 1/2)} & \text{for} & \kappa = -(j+1/2) \end{array}\right. \tag{22}$$

One should pay attention to the possible energy-levels crossings that occur between positive energy sets or negative energy sets. These are unavoidable energy-levels crossings manifested by $\kappa = \pm(j + 1/2)$ and admits the following scenario: A state labeled by Λ_1 and j_1 crosses with a state labeled by Λ_2 and j_2 for all b values when

$$\Lambda_1 + j_1 + \frac{3}{2} = \Lambda_2 - j_2 - \frac{1}{2} \Longrightarrow \Lambda_2 - \Lambda_1 = j_1 + j_2 + 2$$

Nevertheless, quantum numbers related degeneracies are also feasible at different values of b. That is, when

$$\Lambda_1 = \Lambda_2 = \Lambda,\, j_1 \neq j_2 \Longrightarrow \Lambda(b_2 - b_1) = b_1 j_1 + b_2 j_2 + \frac{3b_1 + b_2}{2}$$

and when

$$\Lambda_1 \neq \Lambda_2,\, j_1 = j_2 = j \Longrightarrow \Lambda_2 b_2 - \Lambda_1 b_1 = j(b_1 + b_2) + \frac{3b_1 + b_2}{2}$$

3.3 Dirac-Oscillator-II:

For $A(r) = a/r$ and $M(r) = B^2 r^2/2$ equation (15) reads

$$\left\{-\partial_r^2 + \frac{\tilde{\kappa}(\tilde{\kappa}+1)}{r^2} + B^2\left[E + m\right] r^2\right\}\phi_2(r) = \left[E^2 - m^2\right]\phi_2(r), \tag{23}$$

In this case

$$\left[E^2 - m^2\right] = 2B\sqrt{E+m}\,(2n_r + \tilde{\kappa} + 3/2) \tag{24}$$

which would lead to $E = -m$ (to be discarded) and, with $\tilde{N}_{\pm} = 2n_r + a \pm (j + 1/2) + 3/2 > 0$,

$$E = -m + \left(\frac{1}{3}\xi_{\pm}^{1/3} + 2m\xi_{\pm}^{-1/3} \right)^2 \tag{25}$$

where

$$\xi_{\pm} = 27B\,\tilde{N}_{\pm} + 3\sqrt{-24m^3 + 81B^2\tilde{N}_{\pm}^2}; \quad \tilde{N}_{\pm} \geq \sqrt{\frac{8m^3}{27B^2}} \tag{26}$$

In this case, it is obvious that energy-levels crossings occur when $\xi_+ = \xi_-$. One may, for the sake of simplicity, choose the case where

$$\tilde{N}_{\pm} = \sqrt{\frac{8m^3}{27B^2}} \Longrightarrow \xi_{\pm} = 27B\,\tilde{N}_{\pm},$$

which would imply that a state labeled by n_{r1} and j_1 crosses with a state labeled by n_{r2} and j_2, for all a, when

$$\xi_+ = \xi_- \Longrightarrow \tilde{N}_+ = \tilde{N}_- \Longrightarrow n_{r2} - n_{r1} = (j_1 + j_2 + 1)/2$$

4 Consequences of $A(r) = -\zeta_2'(r)/[2\zeta_2(r)] \neq 0$ and $V(r) \neq M(r)$

If we consider the class of interactions where $A(r) = -\zeta_2'(r)/2\zeta_2(r) \neq 0$ and $V(r) \neq M(r)$, Dirac equation in (11) reduces to

$$\left\{ -\partial_r^2 + \frac{\kappa(\kappa+1)}{r^2} - \zeta_1(r)\zeta_2(r) \right\} \phi_2(r) = 0, \tag{27}$$

which is, in fact, in exact form as that of the radial one-dimensional Klein-Gordon (KG) equation

$$\left\{ -\partial_r^2 + \frac{\ell(\ell+1)}{r^2} - [E - V(r)]^2 + [m + M(r)]^2 \right\} \phi_2(r) = 0. \tag{28}$$

Moreover, it should be noted hereby that the case where $A(r) = -\kappa/r - \zeta_2'(r)/2\zeta_2(r) \neq 0$ and $V(r) \neq M(r)$ corresponds to the s-waves (i.e., $\ell = 0$) solution of (28). Hence, one need not consider it as a separate case to deal with.

4.1 Dirac/Klein-Gordon-Coulomb-II:

For $V(r) = \alpha_1/r$ and $M(r) = \alpha_2/r$ equation (28) reads

$$\left\{ -\partial_r^2 + \frac{\mathcal{L}(\mathcal{L}+1)}{r^2} + \frac{2[\alpha_1 E + \alpha_2 m]}{r} \right\} \phi_2(r) = \left[E^2 - m^2\right] \phi_2(r), \tag{29}$$

with

$$\mathcal{L}=-\frac{1}{2}+\sqrt{(\ell+1/2)^2-\alpha_1^2+\alpha_2^2}\geq 0, \tag{30}$$

and admits exact solution of the form

$$\left[E^2-m^2\right]=-\frac{\left[\alpha_1 E+\alpha_2 m\right]^2}{\mathcal{N}^2};\,\mathcal{N}=n_r+\mathcal{L}+1>0.$$

This would lead to

$$\frac{E_{\pm}}{m}=\frac{-\alpha_1\alpha_2}{\mathcal{N}^2+\alpha_1^2}\pm\left[\left(\frac{\alpha_1\alpha_2}{\mathcal{N}^2+\alpha_1^2}\right)^2+\frac{\mathcal{N}^2-\alpha_2^2}{\mathcal{N}^2+\alpha_1^2}\right]^{1/2}, \tag{31}$$

to yield, for various especial cases of coupling constants,

$$\frac{E_{\pm}}{m}=\begin{cases}\pm\left[1-\alpha_2^2/\mathcal{N}^2\right]^{1/2};\,\mathcal{N}\geq|\alpha_2| & \text{for}\quad \alpha_1=0\wedge\alpha_2\neq 0\\ \pm\left[1+\alpha_1^2/\mathcal{N}^2\right]^{-1/2} & \text{for}\quad \alpha_1\neq 0\wedge\alpha_2=0\\ \left(-\alpha_\circ^2\pm\mathcal{N}^2\right)/\left(\mathcal{N}^2+\alpha_\circ^2\right) & \text{for}\quad \alpha_1=\alpha_2=\alpha_\circ\neq 0\end{cases} \tag{32}$$

It should be obvious that energy-levels crossings for (32) are not feasible at all. At this point, one should notice that the effect of positive/negative κ is absent in the process of choosing $A(r)=-\,\zeta_2'(r)/2\zeta_2(r)\neq 0$ and $V(r)\neq M(r)$. However, the complexification of the coupling constants would manifest *flown-away* states (cf., e.g., [21]) to obtain and disappear from the spectrum. For example, for the case

$$\alpha_1=\alpha_2=i\alpha_\circ\neq 0\Longrightarrow\frac{E_{\pm}}{m}=\frac{\alpha_\circ^2\pm\mathcal{N}^2}{\mathcal{N}^2-\alpha_\circ^2},$$

the energy states *fly-away* and disappear from the spectrum when $\mathcal{N}=|\alpha_\circ|$, and for

$$\alpha_1=i\alpha_1'\neq 0,\alpha_2=0\Longrightarrow\frac{E_{\pm}}{m}=\pm\sqrt{\frac{\mathcal{N}^2}{\mathcal{N}^2-\alpha_1'^2}},$$

energy states *fly-away* when $\mathcal{N}=|\alpha_1'|$. However, *flown-away* states never occur for the case

$$\alpha_1=0,\alpha_2=i\alpha_2'\Longrightarrow\frac{E_{\pm}}{m}=\pm\sqrt{\frac{\mathcal{N}^2-\alpha_2'^2}{\mathcal{N}^2}},$$

but rather the system loses its observability (i.e., reality and discreteness) and collapses when $|\alpha_2'|>\mathcal{N}$. Nevertheless, in connection with such complexified version of α_1 and α_2, the reader may consult Mustafa [21] for more comprehensive details on the spectral properties of a similar complexified Coulombic fields (but with γ in [21] replacing $\mathcal{L}$ in the current example).

4.2 Dirac/Klein-Gordon-Oscillator-III:

For $V(r) = 0$, and $M(r) = \beta_1/r + \beta_2 r - m$ equation (28) implies

$$\left\{ -\partial_r^2 + \frac{\tilde{\mathcal{L}}\left(\tilde{\mathcal{L}}+1\right)}{r^2} + \beta_2^2 r^2 + 2\beta_1\beta_2 \right\} \phi_2(r) = E^2 \phi_2(r),$$

where

$$E_\pm = \pm\sqrt{\beta_2\left(4n_r + 2\tilde{\mathcal{L}} + 3\right) + 2\beta_1\beta_2} \tag{33}$$

with

$$\tilde{\mathcal{L}} = -\frac{1}{2} + \sqrt{(\ell+1/2)^2 + \beta_1^2} \geq 0 \tag{34}$$

Obviously, neither energy-levels crossings nor *flown-away* states are feasible for this model. Nevertheless, one should pay attention to the parametric settings that may lead to imaginary energies and consequently system collapse.

5 Consequences of equally-mixed "*free*"-fields: supersymmetry and quasi-parity

It is obvious that with equally-mixed Lorentz vector and Lorentz scalar "free"-fields (i.e., $V(r) = M(r) = 0$) one would obtain, from (11) and (13),

$$\left\{ -\partial_r^2 + V_\pm(r) \right\} \phi_\pm(r) = \lambda_\pm \phi_\pm(r), \tag{35}$$

where $\lambda_\pm = E^2 - m^2$, $\phi_+(r) = \phi_1(r)$, $\phi_-(r) = \phi_2(r)$, and

$$V_\pm(r) = V_\omega(r) = \tilde{A}(r)^2 + \omega \tilde{A}'(r); \quad \omega = \pm 1$$

stands for an effective supersymmetric-like partner potentials. The supersymmetric language is very well pronounced in this process, therefore. Yet, the *quasi-parity* shall emerge in the forthcoming two Dirac-toy models.

5.1 A Dirac-oscillator-toy

Let us consider a "toy" model:

$$\tilde{A}(r) = -\frac{A}{r} + \frac{1}{2}Br; \ \mathbb{R} \ni A, B > 0,$$

that results in an effective supersymmetric partner "*Dirac-oscillator-toy*" potential of the form

$$V_\omega(r) = \frac{A(A+\omega)}{r^2} + \frac{1}{4}B^2 r^2 - B\left(A - \frac{\omega}{2}\right). \tag{36}$$

In the repulsive/attractive-like core, moreover, one may replace $A(A+\omega)$ by $\sigma(\sigma+1)$. In this case, σ would denote quasi-angular momentum quantum number and $\sigma=-1,0$ may very well correspond to "even" and "odd" *quasi-parity* (i.e., $q=(-1)^{\sigma+1}$), respectively. For more details on quasi-parity convention the reader may refer to, e.g., Mustafa and Znojil [22] and Znojil [23] and related references cited therein. In a straightforward manner, however, one can show that

$$\sigma(\sigma+1)=A(A+\omega)\Longrightarrow\sigma=-\frac{1}{2}+q\left(A+\frac{\omega}{2}\right). \tag{37}$$

Under such settings, it is obvious that both supersymmetric-like partner potentials in (36) admit exact closed form solutions (cf., e.g., Mustafa and Znojil [22], and Znojil [23]):

$$\lambda_\omega=\frac{B}{2}\left(4n_r+2qA+\omega q+2\right)-B\left(A-\frac{\omega}{2}\right);\quad n_r=0,1,2,\cdots, \tag{38}$$

which would split (with $\omega=\pm1$) into

$$\lambda_{+,q} = \frac{B}{2}\left(4n_r+2qA+q+3\right)-BA \Rightarrow\begin{cases}\lambda_{+,q=+1}=2B\left(n_r+1\right)\\ \lambda_{+,q=-1}=2B\left(n_r-A+\frac{1}{2}\right)\end{cases} \tag{39}$$

$$\lambda_{-,q} = \frac{B}{2}\left(4n_r+2qA-q+1\right)-BA \Rightarrow\begin{cases}\lambda_{-,q=+1}=2Bn_r\\ \lambda_{-,q=-1}=2B\left(n_r-A+\frac{1}{2}\right)\end{cases} \tag{40}$$

We may now pay attention to the supersymmetric language "signature" which is documented in the facts that $\lambda_{+,q=-1}=\lambda_{-,q=-1}$, for odd quasi-parity, and $\lambda_{+,q=+1}=\lambda_{-,q=+1}+const.=\lambda_{-,q=+1}+2B$, for even quasi-parity. That is, for an even quasi-parity the superpartner potentials possess identical spectra with a missing lowest state, whereas for an odd quasi-parity the superpartner potentials are "exactly" isospectral. Nevertheless, the consequence of such *"hidden"*-supersymmetry in Dirac equation is very well pronounced in the related Dirac spectra (with $\lambda_\pm=E^2-m^2$):

$$E_{+,q} = \begin{cases}E_{+,q=+1}=+\sqrt{m^2+2B\left(n_r+1\right)}\\ E_{+,q=-1}=+\sqrt{m^2+2B\left(n_r-A+\frac{1}{2}\right)}\end{cases};\ n_r=0,1,2,\cdots \tag{41}$$

$$E_{-,q} = \begin{cases}E_{-,q=+1}=-\sqrt{m^2+2Bn_r}\\ E_{-,q=-1}=-\sqrt{m^2+2B\left(n_r-A+\frac{1}{2}\right)}\end{cases};\ n_r=0,1,2,\cdots \tag{42}$$

In this case, the quasi-parity signature appears in the energy-level crossings between the two sets of energies in (41) and between those in (42). That is, the two sets of energies in (41) cross with each other when

$$E_+\left(n_r=n_{r1},q=+1\right)=E_+\left(n_r=n_{r2},q=-1\right)\Longrightarrow n_{r2}-n_{r1}=A+\frac{1}{2} \tag{43}$$

and those in (42) cross with each other when

$$E_-\left(n_r = n_{r3}, q = +1\right) = E_-\left(n_r = n_{r4}, q = -1\right) \Longrightarrow n_{r4} - n_{r3} = A - \frac{1}{2}. \quad (44)$$

5.2 A Dirac-Coulomb-toy

On the other hand, a "toy" model of the form

$$\tilde{A}\left(r\right) = -\frac{A}{r} + B; \ \mathbb{R} \ni A, B > 0,$$

would result in an effective supersymmetric partner "*Dirac-Coulomb-toy*" potentials

$$V_\omega\left(r\right) = \frac{A\left(A+\omega\right)}{r^2} - \frac{2AB}{r} + B^2, \quad (45)$$

which admits exact solution of the form

$$\lambda_\omega = -\frac{\left(AB\right)^2}{\breve{n}^2} + B^2; \ \breve{n} = n_r + \sigma + 1 > 0. \ n_r = 0, 1, 2, \cdots . \quad (46)$$

This result would split into

$$\lambda_{+,q} = B^2 \left\{ 1 - A^2 \left[n_r + q\left(A + \frac{1}{2} \right) + \frac{1}{2} \right]^{-2} \right\}, \quad (47)$$

$$\lambda_{-,q} = B^2 \left\{ 1 - A^2 \left[n_r + q\left(A - \frac{1}{2} \right) + \frac{1}{2} \right]^{-2} \right\}. \quad (48)$$

Each of which, respectively, yields

$$\lambda_{+,q} \Longrightarrow \begin{cases} \lambda_{+,q=+1} = B^2 \left\{ 1 - A^2 \left[n_r + A + 1 \right]^{-2} \right\} \\ \lambda_{+,q=-1} = B^2 \left\{ 1 - A^2 \left[n_r - A \right]^{-2} \right\} \end{cases} \quad (49)$$

$$\lambda_{-,q} \Longrightarrow \begin{cases} \lambda_{-,q=+1} = B^2 \left\{ 1 - A^2 \left[n_r + A \right]^{-2} \right\} \\ \lambda_{-,q=-1} = B^2 \left\{ 1 - A^2 \left[n_r - A + 1 \right]^{-2} \right\} \end{cases} \quad (50)$$

Yet, it should be noted here that both even and odd quasi-parity eigenvalues $\lambda_{+,q=-1}$ and $\lambda_{-,q=+1}$ allow zero-modes (i.e., zero-energy) at $n_r = 0$ and therefore they do not share the same spectrum (a Nogami's and Toyama's [7] observation). However, $\lambda_{+,q=+1}$ and $\lambda_{-,q=+1}$ do not allow zero-modes and hence they have identical spectra but with a missing lowest state in $\lambda_{-,q=+1}$. Similar

trend is also observed for $\lambda_{+,q=-1}$ and $\lambda_{-,q=-1}$ where the lowest state is missed in $\lambda_{+,q=-1}$. Consequently, the corresponding Dirac spectra read

$$E_{+,q} = \begin{cases} E_{+,q=+1} = +\sqrt{m^2 + B^2 \left\{1 - A^2 \left[n_r + A + 1\right]^{-2}\right\}} \\ E_{+,q=-1} = +\sqrt{m^2 + B^2 \left\{1 - A^2 \left[n_r - A\right]^{-2}\right\}} \end{cases} \tag{51}$$

$$E_{-,q} = \begin{cases} E_{-,q=+1} = -\sqrt{m^2 + B^2 \left\{1 - A^2 \left[n_r + A\right]^{-2}\right\}} \\ E_{-,q=-1} = -\sqrt{m^2 + B^2 \left\{1 - A^2 \left[n_r - A + 1\right]^{-2}\right\}} \end{cases} \tag{52}$$

Evidently, energy-levels crossings obtain between the two sets of energies in (51) or between the two sets of energies in (52). That is, a state $E_{+,q=+1}\,(n_r = n_{r1})$ crosses with a state $E_{+,q=-1}\,(n_r = n_{r2})$ when

$$n_{r1} + A + 1 = n_{r2} - A \Longrightarrow n_{r2} - n_{r1} = 2A + 1,$$

and $E_{-,q=+1}\,(n_r = n_{r3})$ crosses with $E_{-,q=-1}\,(n_r = n_{r4})$ when

$$n_{r3} + A = n_{r4} - A + 1 \Longrightarrow n_{r4} - n_{r3} = 2A - 1.$$

Nevertheless, the energy sets in (51) and (52) loose their reality and become pure imaginary in the following manner:

$$\begin{aligned} E_{+,q=+1} &\in \mathbb{C} \text{ for } m^2 + B^2 < A^2B^2/\left(n_r + A + 1\right)^2, \\ E_{+,q=-1} &\in \mathbb{C} \text{ for } m^2 + B^2 < A^2B^2/\left(n_r - A\right)^2, \\ E_{-,q=+1} &\in \mathbb{C} \text{ for } m^2 + B^2 < A^2B^2/\left(n_r + A\right)^2, \\ E_{-,q=-1} &\in \mathbb{C} \text{ for } m^2 + B^2 < A^2B^2/\left(n_r - A + 1\right)^2. \end{aligned}$$

6 Conclusion

The inspiration of the current work is stimulated by our subsequent study [20] of the hidden/built-in supersymmetric quantum mechanical language and/or quasi-parity signatures on the spectrum of the (1+1)-Dirac equation. However, as long as Dirac and Klein-Gordon wave equations are concerned, the energy-levels crossing phenomenon/paradox (the discussion of which already lies far beyond our current proposal, cf., e.g., [19] for more details) as a spectral property or as a consequence of the supersymmetric language and/or quasi-parity is left an *almost-forgotten* one. Our purpose, even with the current over-simplified radial Dirac/Klein-Gordon examples, was to fill this gap at least partially.

In the light of the current study, we have observed that under different settings of the magnetic interaction field $A(r)$, and likewise the related interactions' way-of-coupling into Dirac equation, the two ultimate/effective descendents, Dirac-Coulomb and Dirac-oscillator, perform energy-levels crossing at

different conditions. Moreover, our observations in section 5 on the spectral properties of the radial supersymmetric partner Hamiltonians (i.e., $\lambda_{\pm,q=\pm1}$ in (39), (40), (49), and (50)) re-confirm Nogami's and Toyama's [7] ones on the (1+1)-dimensional Dirac equation.

Finally, we contemplate that the variety of settings (presented in this work) of the related interactions' way-of-coupling (i.e., Eqs. (15), (27), and (35)) into the (3+1)-dimensional radial Dirac equation would enrich the number of exactly/quasi-exactly/conditionally-exactly solvable Dirac models. Yet, the solution of the most general radial case in (13) still resides in the mathematically challenging Hermitian, non-Hermitian, and pseudo-Hermitian [20,21,24-26] Dirac territories.

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